EXACT SOLUTIONS OF THE PROBLEM OF THE MOTION OF THE INTERFACE BETWEEN IMMISCIBLE FLUIDS IN A POROUS MEDIUM

V. V. Voinov

UDC 532.546

In this article, we examine the steady motion of two fluids in a porous medium with a stationary interface. We obtain exact solutions to plane and axisymmetric problems of the motion of a drop. The laws governing the motion of the drop are linked with the mechanism of decay of the displacement front at the nonlinear stage.

As is known, in the displacement of a viscous fluid from a porous medium by a fluid having a lower viscosity, the interface between the two fluids is unstable. Small perturbations of the interface grow rapidly and assume the form of fingers at the nonlinear stage. Periodic solutions of the plane problem were examined in [1-3]. Tryggvason and Aref [4, 5] numerically studied the evolution of small periodic perturbations of a plane interface, assigning the disturbances in the form of superimposed perturbations of different periods. The investigators not only obtained fingers in the nonlinear stage, but they also observed their breakup into drops.

It is interesting to attempt to analytically describe a drop and single fingers moving in a porous medium.

1. Formulation of the Problem. A drop composed of a given fluid is moving in a viscous fluid of a different nature. The viscous fluid is being filtered in an infinite, uniform, porous medium. The fluid comprising the drop is bounded by the surface S. The potential Φ of the filtration velocity $\mathbf{V}$ satisfies the Laplace equation

$$\Delta\Phi_{1,2} = 0. \tag{1.1}$$

Here and below, the subscript 1 denotes the region outside the drop, while 2 denotes the internal region. Away from the surface S of the drop, we assign a constant filtration velocity

$$\nabla\Phi_1 \to \mathbf{V}_\infty,\ r \to \infty. \tag{1.2}$$

Conditions of impermeability of the boundary S to the flow, with the boundary moving at the velocity $\mathbf{V}_0$, yield two boundary conditions for the surface of the drop:

$$\left.\frac{\partial\Phi_1}{\partial n}\right|_S = \left.\frac{\partial\Phi_2}{\partial n}\right|_S = (\mathbf{V}_0\mathbf{n}). \tag{1.3}$$

The condition of continuity of the pressure at the boundary leads to a third boundary condition which includes the absolute viscosities μ_1, μ_2:

$$\mu_1\Phi_1|_S = \mu_2\Phi_2|_S + \text{const}. \tag{1.4}$$

Problem (1.1)-(1.4) is a steady-state problem with an unknown free boundary S containing the velocity of the drop V_0 as a parameter.

The internal problem for Φ_2, with boundary condition (1.3), has a unique solution — the potential of the uniform flow $\Phi_2 = V_0x$, where x is the coordinate in the direction of motion of the drop (or z).

An attempt can be made to find exact solutions to the problem of a drop by examining bounding surfaces that can be described by second-order equations. Here, it is natural to make use of the corresponding orthogonal coordinates u, v. In these coordinates, the boundary condition for Φ_1 (1.3) is written in the form

Moscow. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 68-71, January-February, 1991. Original article submitted November 25, 1988; revision submitted September 22, 1989.

 0021-8944/91/3201-0064$12.50

$$\left.\frac{\partial\Phi_1}{\partial v}\right|_{v=v_0} = \left.\frac{\partial\Phi_2}{\partial v}\right|_{v=v_0}. \tag{1.5}$$

Here, $v = v_0$ is the equation of the surface of the drop S. For the sake of simplicity, we will henceforth examine the situation in which filtration velocity away from the drop $\mathbf{V}_\infty$ and the velocity of the drop $\mathbf{V}_0$ are colinear.

2. Plane Problem of the Motion of a Parabolic Drop. If we attempt to solve the plane problem of a drop with a parabolic contour, it turns out to be convenient to make use of parabolic coordinates u, v. In these coordinates, Eq. (1.1) has the same form that it does in Cartesian coordinates.

The potential of the external flow Φ_1 consists of the sum of the potential of the uniform flow and a potential of the form βv, where the constant β is determined by condition (1.5): $\Phi_1 = V_\infty x - (V_0 - V_\infty)v_0 v$. Inserting the expressions for the potentials of the external and internal flows into boundary condition (1.4), we find the velocity of a parabolic drop

$$V_0 = V_\infty \mu_1/\mu_2. \tag{2.1}$$

The above solution corresponds to the limiting situation in the problem of the motion of a bubble between two solid walls [1], when the ratio of the width of the bubble to the width of the channel approaches zero.

3. Motion of an Axisymmetric Paraboloid of Filtering Fluid. We will examine an axisymmetric problem concerning the motion of a displacement front of parabolic form. The solution in the region external to the paraboloid has the form

$$\Phi_1 = V_\infty z - (V_0 - V_\infty)v_0^2 \ln v.$$

Dynamic boundary condition (1.4) is satisfied if the filtration velocity inside the paraboloid is determined by (2.1), i.e., if it coincides with the velocity of a plane parabolic drop. It is interesting that the velocity of a paraboloid of filtering fluid is independent of the curvature at its vertex. The solution is valid for a one-parameter family of paraboloids.

4. Plane Elliptical Drop. To describe the motion of an elliptical drop, it is convenient to use elliptical coordinates. In the plane case, this includes cofocal u-hyperbolas and v-ellipses [6]. The solution is sought in a manner similar to that described above, as the superposition of the potential of the unidimensional flow on the one hand and, on the other hand, the potentials $\beta \sin u \exp(-v)$ and $\beta \cos u \exp(-v)$ for prolate and oblate ellipses. Using (1.5) to find the coefficient β, we obtain expressions for Φ_1 in the cases of prolate and oblate ellipses:

$$\Phi_1 = \alpha \sin u(V_\infty \operatorname{ch} v + (V_\infty - V_0) \operatorname{sh} v_0 \exp(v_0 - v)),$$
$$\Phi_1 = \alpha \cos u(V_\infty \operatorname{sh} v + (V_\infty - V_0) \operatorname{ch} v_0 \exp(v_0 - v)).$$

We use condition (1.4) to find the velocity ratio

$$V_0/V_\infty = (1 + g)/(\lambda + g),\ \lambda = \mu_2/\mu_1. \tag{4.1}$$

Here, $g = \chi$ for a prolate ellipse and $g = \chi^{-1}$ for an oblate ellipse; $\chi = \sqrt{1 - \delta^2}$ is the ratio of the semiaxes; $\delta = \cosh^{-1} v_0$ is the eccentricity. It can be seen from (4.1) that the velocity of a plane elliptical drop depends considerably on the degree of its deformation.

5. Motion of an Ellipsoidal Drop. Let a drop of fluid occupy the volume of an ellipsoid of revolution. Proceeding analogously to Part 4, we will use u-hyperboloids and v-ellipsoids as the coordinate surfaces. The form of the Laplace equation for these coordinates was presented in [6]. The potential of the external flow Φ_1 is sought in the form of the superposition of the potential of the unidimensional flow $V_\infty z$ and a potential of the form $\sin(u)(c_2 Q_1(\cosh v) + c_1 \cosh v)$ for a prolate ellipsoid and $\cos(u)(c_2 Q_1(i \sinh v) + c_1 \sinh v)$ for an oblate ellipsoid ($Q_1(x)$ is a second-order Legendre function with the index 1). The constant c_1 is determined on the basis of the condition that the function be real and be bounded at $v \to \infty$. After this, the potential is expressed for prolate and oblate el-

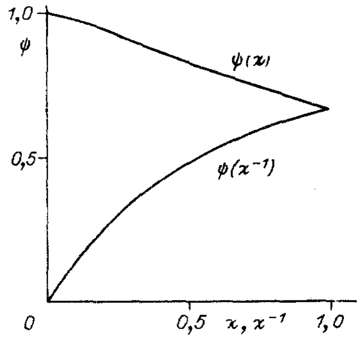

Fig. 1

lipsoids by means of the formulas $\Phi_1 = \alpha \sin u(V_\infty \cosh v + \beta(\cosh v \times \operatorname{arctanh}(\cosh v) - 1))$ and $\Phi_1 = \alpha \cos u(V_\infty \sinh v + \beta(\sinh v \operatorname{arccot}(\sinh v) - 1))$. Satisfying boundary conditions (1.4)-(1.5), we find the velocity ratio and β: $V_\infty/V_0 = 1 + (\lambda - 1)\psi_{+,-}$, $\beta = g_1(\lambda - 1)V_0 \sinh 2v_0/2$. Here, we have the following for a prolate ellipsoid of revolution

$$\psi_+ = \delta^{-2} - (\delta^{-2} - 1)\delta^{-1}\mathrm{Arth}\ \delta,\ g_1 = \mathrm{sh}\ v_0, \tag{5.1}$$

while for an oblate ellipsoid of revolution

$$\psi_- = 1 - \delta^{-2} + \delta^{-3}\sqrt{1-\delta^2}\ \mathrm{arctg}\ (\delta/\sqrt{1-\delta^2}),\ g_1 = -\mathrm{ch}\ v_0. \tag{5.2}$$

When the eccentricity of the ellipsoid approaches unity – with the prolate ellipsoid being transformed into a "needle" and the oblate ellipsoid into a disk – the formulas (5.1)-(5.2) for the velocity coefficient ψ take the form

$$\psi_+ \approx 1 + (1-\delta)\ln(1-\delta),\ \psi_- \approx (\pi/\sqrt{2})\sqrt{1-\delta}. \tag{5.3}$$

It follows from this that an oblate ellipsoid of revolution moves at nearly the same velocity at which the surrounding fluid is being filtered, while a prolate ellipsoid moves at a velocity which is greater by a factor of μ_1/μ_2. In the case of small eccentricity, Eqs. (5.1)-(5.2) yield

$$\psi_+ \approx (2/3)(1 + \delta^2/5),\ \psi_- \approx (2/3)(1 - \delta^2/5). \tag{5.4}$$

If $\lambda = \mu_2/\mu_1 < 1$, then the oblate drop will move more slowly than the prolate drop. For plane drops with small δ, formulas analogous to (5.4) follow from (4.1)-(4.2). Meanwhile, $V_\infty/V_0 = (\lambda + 1)/2$. For highly prolate or oblate plane drops, when δ = 1, Eqs. (4.1)-(4.2) give the same velocities as (5.3).

The dependence of the velocity coefficient of an ellipsoidal drop on the degree of its deformation ψ(χ) is equal to the ratio of the semiaxes of the ellipsoid (the semiaxis along the axis of revolution – in the denominator), is shown in Fig. 1. It is evident from the figure that drop velocity is heavily dependent on the deformation of the drop. It is interesting that a highly prolate drop moves at the same velocity as a parabolic "tongue," in accordance with (2.1), (4.1), and (5.1).

If $\mu_2/\mu_1 > 1$, then the velocities of both oblate and prolate drops are lower than the fluid velocity at infinity. Thus, they converge with the displacement front if they are ahead of it and lag behind the front if they are behind it. The displacement front is obviously stable in the first case and unstable in the second case.

It is significant that the velocity of a prolate elliptical drop at λ > 1 is λ times lower than the fluid velocity at infinity. Thus, in order to have a certain volume of viscous (relative to water) oil be displaced by water, a substantial amount of water will have to be pumped if the oil is distributed in the form of prolate drops.

It is also interesting that – as follows from the above formulas – prolate drops with a small internal viscosity $\mu_2 < \mu_1$ overtake less prolate drops. Meanwhile, highly oblate drops move slowest of all. Thus, the most stable prolate drops move ahead of the displacement front. This effect was observed in photographs in the experimental study [7], where a prolate drop catches up with a less prolate drop and combines with it. The opposite effect is seen for the more viscous fluid which lags behind the displacement front. Here, the most prolate drops lag the farthest behind the front.

LITERATURE CITED

1. P. G. Saffman and G. Taylor, "The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous fluid," Proc. R. Soc., 245A, No. 1242 (1958).
2. Sir Geoffrey Taylor and P. G. Saffman, "A note on the motion of bubbles in a Hele-Shaw cell and porous medium," Q. J. Appl. Mech. Math., XII, Pt. 3, No. 8 (1959).
3. P. G. Saffman, "Viscous fingering in Hele-Shaw cells," J. Fluid Mech., 173, No. 12 (1986).
4. G. Tryggvason and H. Aref, "Numerical experiments on Hele-Shaw flow with a sharp interface," ibid., 136, No. 11 (1983).
5. G. Tryggvason and H. Aref, "Finger-interaction mechanisms in stratified Hele-Shaw flow," ibid., 154, No. 5 (1985).
6. Oe. Madelung, Mathematical Physics [Russian translation], Nauka, Moscow (1968).
7. T. Maxworthy, "Bubble formation, motion, and interaction in Hele-Shaw cell," J. Fluid Mech., 173, No. 12 (1986).

NONSTEADY THERMAL CONVECTION IN A HORIZONTAL CYLINDER WITH A NONUNIFORM DISTRIBUTION OF THE TEMPERATURE OF THE BOUNDARIES

B. G. Petrazhitskii and N. M. Stankevich UDC 536.25

Many studies have examined natural convection in a horizontal cylinder. Nonsteady natural convection in closed rectangular and spherical cavities was examined in [1, 2]. A survey of different studies can be found in [3], while the latest publications are discussed in [4]. Some of these investigations consider the effect of a nonuniform distribution of the temperature of the boundaries. In particular, Gershuni et al. [5] and Ostrakh et al. [6] used approximate analytical methods to obtain information on local and integral characteristics of the given phenomenon in a steady-state regime. The authors made several simplifying assumptions that limited the range of application of the results. Thus, the data in [5] was obtained in a boundary-layer approximation, while the data in [6] is valid only for large Prandtl numbers and for Grashof numbers on the order of unity. The range of phase angles corresponding to cosine distributions of boundary temperature $0 < \varphi < \pi/2$ (Fig. 1), except for the region of the points $\varphi = 0$ (heating from the side) and $\varphi = \pi/2$ (heating from below).

A numerical solution to the problem was described in [7] for $-\pi/2 < \varphi < \pi/2$. Extensive information was presented on the streamlines and isotherms for different φ and $Pr = 1$. In contrast to the present study, the results in [7] pertain only to the steady-state regime and contain no information on velocity and temperature fields or local heat-transfer characteristics.

We attempted to solve the complete system of Navier–Stokes equations by numerical methods to obtain data on the process of establishment of a steady-regime and fill in the missing data for it for the range $10^4 < Ra < 10^7$ and $Pr = 0.68$ (helium) at $-\pi/2 < \varphi \le 0$.

The given phenomenon has several important practical applications and is described by the system of equations of motion, continuity, and energy. We will study the two-dimensional laminar flow of an incompressible fluid with constant physical properties and a linear temperature dependence of density. After we exclude pressure and introduce the stream function ψ, this system takes the following form in polar coordinates (r, θ) for the conditions of the given problem (ω is curl and Φ is temperature):

Gorky. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 1, pp. 71-77, January-February, 1991. Original article submitted January 23, 1988; revision submitted June 26, 1989.

